Quantum Information Metrics and Relative Entropies
– Classification Problem –

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対称性と正値写像に対する単調減少性を備えたリーマン距離としての量子情報距離と量子相対エントロピーとの関係を述べ、それに基き Wigner-Yanase-Dyson 情報量の普遍性について論ずる。

Quantum information metric is a Riemannian metric defined on matrix spaces which measures an infinitesimal distance between two nearby density matrices ρ and $\rho + d\rho$ whose classical correspondent is the so-called Fisher information metric. In the classical version, this concept is universal among all those having monotonicity against coarse-grainimg(Chentsov 1982), and is derived by twice differentiations of an entropy functional, called α-divergence(Amari 1985), such that

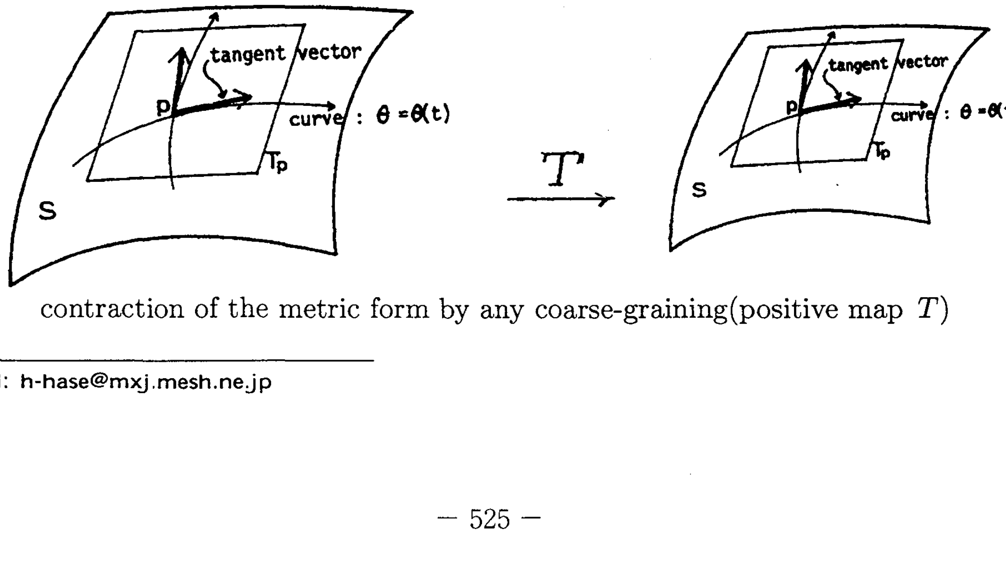
$$K^F_{ij}(\theta) = \int \partial_i \log p(x,\theta) \partial_j \log p(x,\theta) p(x,\theta) d\mu(x) = \langle \partial_i \log p(\theta) \partial_j \log p(\theta) \rangle = -\partial_i \partial'_j S_{g_\alpha}(p(\theta), p(\theta'))|_{\theta'=\theta}, \quad \text{where} \tag{1}$$

$$S_{g_\alpha}(p,q) \equiv \frac{4}{1-\alpha^2} \int \left(1 - q(x,\theta)^{\frac{1+\alpha}{2}} p(x,\theta)^{\frac{-(1+\alpha)}{2}} \right) p(x,\theta) d\mu(x) \tag{2}$$

(α-divergence for two probabilities $p(x, \{\theta_i\}), q(x, \{\theta_i\})$).

Thus, the said universality is evidenced by the fact that K^F is α-independent. Unlike the classical context,quantum information metrics with monotonicity are not universal in the above sense, and we aim to get a relevant universality with an aid of relative or quasi-entropy(Petz, 1985).

Fig.1 Image of a monotone Riemannian metric.



contraction of the metric form by any coarse-graining(positive map T)

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1 Outline of Results– with set of metrics and quasi-entropies

set of all Petz's symmetric monotone functions (1996) $f \in \mathcal{F} \supset \mathcal{F}_{power}$ and $\mathcal{F}_{WYD}$
("power"denotes power-mean; and "WYD" Wigner-Yanase-Dyson, for monotone metrics)

Result(1): $\mathcal{F}_{power} \cup \mathcal{F}_{WYD}$ is dense in $\mathcal{F}$ by a properly defined norm topology.

set of all convex g-functions Petz(1985) $g \in \mathcal{G} \supset \mathcal{G}_{sym}, \mathcal{G}_{asym}$ ($\mathcal{G}_{sym}, \mathcal{G}_{asym}$ with and without satisfying condition of symmetry $g(x) = g^{dual}(x) \equiv xg(x^{-1})$ and $g(x) \neq g^{dual}(x)$, respectively). Correspondence between $\mathcal{G}$ and $\mathcal{F}$ in **Sec.3**(Lesniewski-Ruskai(1999)) yields

Result(2): there exists no g-function in $\mathcal{G}_{asym}$ other than quantum α-divergence: it is the only source of WYD metrics that possess the potential duality of geometry.

Historical remark: the Wigner-Yanase-Dyson metrics

The original definition of skew information was $-\frac{1}{2}\mathrm{Tr}[\rho^{1/2}, k]^2$ (Wigner-Yanase in Proc. Nat. Acad. Sc., 1963), to which Dyson suggested that it could be generalized to the one exponent ϑ and the other $1-\vartheta$ for the power of ρ. Lieb(1973), substantiating their convexity for the first time, expressed them as

$$I_p(\rho, \Delta) \equiv \frac{1}{p(1-p)}\mathrm{Tr}[\rho^p, \Delta][\rho^{1-p}, \Delta] \quad (\Delta \in \{\text{skew hermitians}\},\ i\Delta \in \mathcal{M}_h)$$
$$= \frac{4}{1-\alpha^2} D_\sigma^\perp D_\rho^\perp \sigma^{\frac{1+\alpha}{2}}(A)\rho^{\frac{1-\alpha}{2}}(A))|_{\sigma=\rho} \text{ with } D_\rho^\perp(\varphi(\rho))(A) \equiv [\varphi(\rho), \Delta_A] \quad p \equiv \frac{1+\alpha}{2}.$$

This adds to the classical Fisher term to yield a full metric form in terms of a Fréchet differential denoted by D_ρ: a Fréchet derivation on a matrix function $\varphi(\rho)$ is written as $D_\rho\varphi(\rho)(A)$ with A being an arbitrary hermitian except $\mathrm{Tr}A = 0$ a matrix tangent vector, illustration in Fig.1 with an important relation $A = [\varphi(\rho), \Delta_A]$ (Δ_A only depends on A), leading us to

$$\mathrm{Tr}A^{c\dagger}g^F A^c + I_p(\rho, \Delta_A)\,(g^F = \rho^{-1}) = \mathrm{Tr}D_\rho(p^{-1}\rho^p)(A)D_\rho((1-p)^{-1}\rho^{1-p})(A). \tag{3}$$

The quantum correspondent to the Fisher metric has an additional term, as indicated above in eq.(3), which could be derived from the corresponding quantum α-divergence. Namely,

$$S_\alpha(\rho, \rho+\rho)(\text{to 2nd order in } d\rho\ ) = \frac{1}{2}\mathrm{Tr}\rho d^c \log\rho d^c \log\rho + \frac{4}{1-\alpha^2}\mathrm{Tr}[\rho^{\frac{1+\alpha}{2}}, \Delta][\rho^{\frac{1-\alpha}{2}}, \Delta]\ , \tag{4}$$

where symbol d^c represents the derivative of ρ which commutes with ρ: it means that an infinitesimal increment of a noncommutative variable should consist in a commutative and a noncommutative parts with ρ(former yielding the uniform Fisher term, whereas the latter nonuniform)

$$d\rho = d^c\rho + [\rho, \Delta] \quad [d^c\rho, \rho] = 0 \quad \text{and } \Delta \in \{\text{skew hermitians}\} or\ i\Delta \in \mathcal{M}_h.$$

This may be regarded as a Hilbert-Schmidt orthogonal decomposition, since $\mathrm{Tr}d^c\rho[\rho, \Delta] = 0$ holds. It is then easy to see the following expression $d\varphi(\rho)$ for any analytic function of ρ $\varphi(\rho) = \frac{1}{2\pi i}\int_C \frac{dz}{z-\rho}$; quantum analog of tangent vector at the point ρ as illustrated in Fig.1 i.e.

$$d\rho = d^c\rho + [\rho, \Delta]; \qquad d\varphi(\rho) = \varphi'(\rho)d^c\rho + [\varphi(\rho), \Delta].$$

2 General monotone metrics(extended Chentsov theorem)

for including Bures metric and for interpolating it with the WYD by the power-mean metrics:

$$K_\rho(A,B) \equiv \mathrm{Tr} A \mathbf{K}_\rho B \qquad A,B \in \mathcal{M}_h, \text{ and for symmetric-metric cases}$$
$$K'_\rho(B,A) \equiv \frac{1}{2}(K_\rho(B,A) + K_\rho(A,B)) = K'_\rho(A,B); \text{ redenoted by } K_\rho(A,B).$$

By taking the ρ-diagonalized basis, it can be expressed in terms of a two-variable function $c(\lambda,\mu)$ on $\mathbf{R}^+ \times \mathbf{R}^+$ and a single variable one $c(\lambda) \equiv c(\lambda,\lambda)$ on $\mathbf{R}^+$ in the above bilinear form:

$$K_\rho(B,A) = \sum_i c(\lambda_i) B_{ii} A_{ii} + 2 \sum_{i<j} c(\lambda_i,\lambda_j) B^*_{ij} A_{ij} \qquad \text{with}$$

$c(x\lambda, x\mu) = x^{-1} c(\lambda,\mu)$ (homogenuity of MC-functions of order -1), in particular $c(\lambda,\lambda) = \lambda^{-1}$.

Petz's representation of symmetric monotone metrics by f:

$K_\rho(B,A) = \langle A, R_\rho^{-1/2} f(L_\rho R_\rho^{-1})^{-1} R_\rho^{-1/2} B\rangle$, where $f(x)$ and $c(\lambda,\mu)$ are related to each other:

$$f(x) = \frac{1}{c(x,1)} \quad \text{or} \quad c(\lambda,\mu) = \frac{1}{\mu f(\lambda/\mu)}. \tag{5}$$

Representation of symmetry $c(\mu,\lambda) = c(\lambda,\mu) \Leftrightarrow x f(1/x) = f(x)$. Then, $c(\lambda = \mu) = 1/\lambda \rightarrow f(1) = 1$. In classical information geometry, the g-divergence function conditioned by $g(1) = 0$ and $g''(x) > 0, (= 0$ only for $x = 1)$, was shown to be entitled all the way to yield α-divergence with any real number $\alpha(|\alpha| = 3 + g'''(1)/g''(1)$ Amari,1986). One of remarkable outcomes of the extended Chentsov theorem is the limited order structure of the monotone metrics so that

$$(\text{Bures})\frac{1+x}{2} = f_{max}(x) \geq f_{power}(x) \geq f_{WYD}(x) \geq f_{min}(x) = \frac{2x}{1+x}(\text{WYD}^{|\alpha|=3}). \tag{6}$$

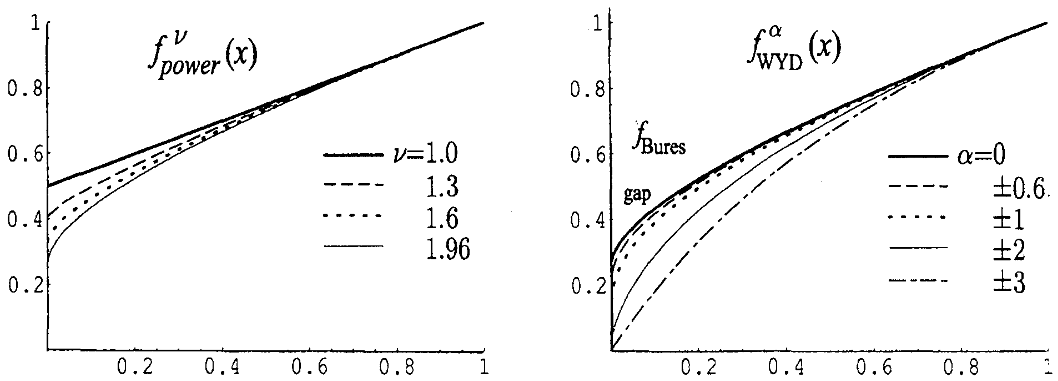
Specific feature of quantum information metrics There appears a region which may be called "gap region" between $f_{max} = f_{Bures}$ and $f^{\alpha=0}_{WYD}$. This is absent in the classical manifold.

Fig.2 Order structure of monotone metrics on matrix spaces in terms of f-functions.

the "gap" region **the WYD region**

$$f^\nu_{power}(x) = \left(\frac{1+x^{1/\nu}}{2}\right)^\nu \quad 1 \leq \nu \leq 2; \quad f^\alpha_{WYD}(x) = \frac{1-\alpha^2}{4} \frac{(1-x)^2}{(1-x^{\frac{1-\alpha}{2}})(1-x^{\frac{1+\alpha}{2}})} \quad |\alpha| \leq 3. \tag{7}$$

($\mathcal{F}_{power} \cup \mathcal{F}_{WYD}$ forms a linear-ordered subset of $\mathcal{F}$ with coincident min-max bounds)



3 Classification of quasi-entropies by symmetry

We follow Lesniewski-Ruskai(J. Math. Phys.**40**(1999)5702-5723) to represent a convex operator functiong(x)in $S_g(\rho,\sigma)=\mathrm{Tr}(\sigma-\rho)g^{(2)}(L_\sigma R_\rho^{-1})R_\rho^{-1}(\sigma-\rho)$; $g^{(2)}(x)\equiv\dfrac{g(x)}{(x-1)^2}$, where $g(x)=b(x-1)^2+c\dfrac{(x-1)^2}{x}+\displaystyle\int_0^\infty\frac{(x-1)^2}{x+s}dm(s)$, two constants $b,c\geq 0$ and a measure $m\geq 0$. (8)

We define, for a $g(x)$, $g^{dual}(x)\equiv xg(x^{-1})$. Then, $((x-1)^2)^{dual}=(x-1)^2/x$ and vice versa. In general,

$$g^{dual}(x)=c(x-1)^2+b\frac{(x-1)^2}{x}+\int_0^\infty\frac{(x-1)^2}{x+s}d\bar{m}(s),\quad \text{where } \bar{m}(s)\equiv sm(s^{-1}). \tag{9}$$

Let $\mathcal{G}_{sym}$ and $\mathcal{G}_{asym}$ denote the set of all quasi-entropy g-functions of x, defined for symmetric and asymmetric class, respectively, by $\mathcal{G}_{sym}=\{g;g(x)=g^{dual}(x)\}$ and $\mathcal{G}_{asym}=\{g;g(x)\neq g^{dual}(x)\}$.

(1) Selfdual class: $S_{s.dual}(\rho,\sigma)=S_{s.dual}(\sigma,\rho)$ with a symmetric $g\in\mathcal{G}_{sym}$. In terms of the measure representations above, both $b=c$ and $\bar{m}(s)=m(s)$ hold.

(2) Non-selfdual class: $S_{ns.dual}(\rho,\sigma)\neq S_{ns.dual}(\sigma,\rho)$ with an asymmetric $g\in\mathcal{G}_{asym}$. Then,

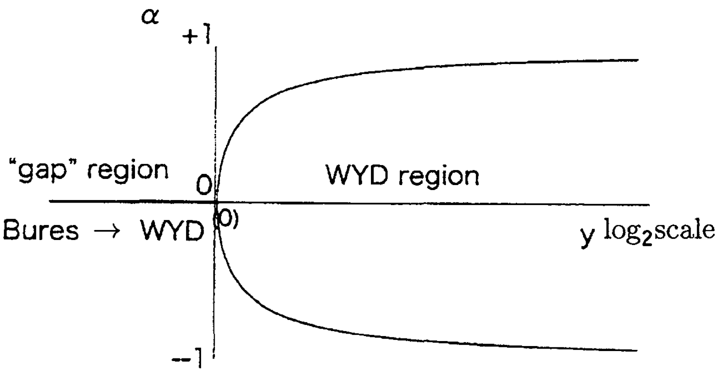
Lesniewski-Ruskai relation between Petz's $f(x)$ and $g(x)$ holds: $\dfrac{1}{f(x)}=\dfrac{g(x)+g^{dual}(x)}{2(x-1)^2}$. (10)

4 Concluding remarks

4.1 transition from the classical Fisher to quantum information metrics

Fig.3 Schematic picture: appearance of the quantum structure of information metrics.

Plot $\alpha\, vs\, y=f(x=0+)^{-1}$ in (7).
It is reminiscent of bifurcation
along the transition line: (Fisher) → Bures → WYD



More studies are necessary for succeeding in this idea.

4.2 universality of the WYD metrics endowed by an inclusion of topology

Lesniewski-Ruskai relation (10) says that, if a symmetric monotone metric f is identical to a member of the WYD class, then the associated quasi-entropy g-function must be equal to the $\pm\alpha$-divergence $g_{\pm\alpha}$[1]. However, it does not assure that $g_{\pm\alpha}$'s exhaust all members of $\mathcal{G}_{asym}$. We have investigated this question, introducing a norm topology by exploiting the symmetry of f, which makes it equivalent to the space $C[0,1]$(set of all continuous functions on $[0,1]$). Consequently, the question can be answered affirmatively. Details will be given elsewhere.

All citations in the present article are given in

[1] H. Hasegawa,Infinite Dimensional Analalysis, Quantum Probability and Related Topics **6** (2003),413-430; **数理科学**「ランダム行列理論と非可換情報幾何」長谷川 洋 (現代物理と現代幾何 2002),105-114 および「量子情報理論と幾何」古池達彦 (**488** 2004),15-21 サイエンス社.